

# PLURIPOTENTIAL THEORY AND CONVEX BODIES

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**ABSTRACT.** In their seminal paper [4], Berman and Boucksom exploited ideas from complex geometry to analyze asymptotics of spaces of holomorphic sections of tensor powers of certain line bundles  $L$  over compact, complex manifolds as the power grows. This yielded results on weighted polynomial spaces in weighted pluripotential theory in  $\mathbb{C}^d$ . Here, motivated from [1], we work in the setting of weighted pluripotential theory arising from polynomials associated to a convex body in  $(\mathbb{R}^+)^d$ . These classes of polynomials need not occur as sections of tensor powers of a line bundle  $L$  over a compact, complex manifold. We follow the approach in [4] to recover analogous results.

## 1. INTRODUCTION

Motivated by probabilistic results in [1] as well as some questions in multivariate approximation theory [9], we study pluripotential-theoretic notions associated to closed subsets  $K \subset \mathbb{C}^d$  and weight functions  $Q$  on  $K$  in the following setting. Given a convex body  $P \subset (\mathbb{R}^+)^d$  we define finite-dimensional polynomial spaces

$$Poly(nP) := \{p(z) = \sum_{J \in nP \cap (\mathbb{Z}^+)^d} c_J z^J : c_J \in \mathbb{C}\}, \quad n = 1, 2, \dots$$

associated to  $P$ . Here  $z^J = z_1^{j_1} \cdots z_d^{j_d}$  for  $J = (j_1, \dots, j_d)$ . The main goal of this work is to give a self-contained presentation of some of the results and techniques of R. Berman, S. Boucksom and D. Nystrom in [4] and [5], valid in the setting of holomorphic sections of tensor powers of certain line bundles  $L$  over compact, complex manifolds, for the spaces  $Poly(nP)$ . A key result in [4] relates asymptotics of ball volume ratios of spaces of holomorphic sections with an Aubin-Mabuchi

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type energy of appropriate pluripotential-theoretic extremal functions. Our spaces  $Poly(nP)$  do not generally arise as holomorphic sections of tensor powers of a line bundle. However, many of the techniques in [4] and [5] are available and we are able to modify their approach to prove the analogous key result, Theorem 5.1, and similar consequences; e.g., that *asymptotically weighted  $P$ -Fekete arrays* and *weighted  $P$ -optimal measures* distribute asymptotically like the Monge-Ampere measure  $(dd^c V_{P,K,Q}^*)^d$  of the weighted  $P$ -extremal function (Corollaries 6.5 and 6.4). A difference with [4] and [5] is that here we deduce the existence of a weighted  $P$ -transfinite diameter; i.e., a limit of scaled maximal weighted Vandermondes, as a consequence of Theorem 5.1 (see Remark 5.2).

In the next section, we give definitions and background for the relevant pluripotential-theoretic notions. We define Lelong classes  $L_P$  and  $L_{P,+}$  associated to a convex body  $P \subset (\mathbb{R}^+)^d$ . For certain  $K \subset \mathbb{C}^d$  and  $Q : K \rightarrow \mathbb{R}$  we define a weighted  $P$ -extremal function  $V_{P,K,Q}$ ; weighted  $P$ -transfinite diameter, and weighted  $P$ -optimal measures. Ball volume ratios, as defined and utilized in [4], are discussed in subsection 2.5. In section 3 we discuss the Aubin-Mabuchi type energy  $\mathcal{E}(u, v)$  associated to a pair of functions  $u, v$  in  $L_{P,+}$ . The differentiability of the composition of  $\mathcal{E}$  with a projection operator, proved in section 4, is a key step in verifying the main result, Theorem 5.1, on ball volume ratio asymptotics. This latter is proved in section 5. Both sections follow arguments in [4]. The applications described in the previous paragraph are given in section 6, following [5].

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## 2. BACKGROUND.

2.1.  **$P$ -extremal functions: Results from [1].** Let  $\mathbb{R}^+ = [0, \infty)$ . We fix a *convex body*  $P \subset (\mathbb{R}^+)^d$ ; i.e.,  $P$  is compact, convex and  $P^\circ \neq \emptyset$ . A standard example occurs when  $P$  is a non-degenerate convex polytope, i.e., the convex hull of a finite subset of  $(\mathbb{Z}^+)^d$  in  $(\mathbb{R}^+)^d$  with nonempty interior. Associated with  $P$ , following [1], we consider the finite-dimensional polynomial spaces

$$\text{Poly}(nP) := \{p(z) = \sum_{J \in nP \cap (\mathbb{Z}^+)^d} c_J z^J : c_J \in \mathbb{C}\}$$

for  $n = 1, 2, \dots$  where  $z^J = z_1^{j_1} \cdots z_d^{j_d}$  for  $J = (j_1, \dots, j_d)$ . We let  $d_n$  be the dimension of  $\text{Poly}(nP)$ . For  $P = \Sigma$  where

$$\Sigma := \{(x_1, \dots, x_d) \in \mathbb{R}^d : 0 \leq x_i \leq 1, \sum_{j=1}^d x_j \leq 1\},$$

we have  $\text{Poly}(n\Sigma) = \mathcal{P}_n$ , the usual space of holomorphic polynomials of degree at most  $n$  in  $\mathbb{C}^d$ . Given  $P$ , there exists a minimal positive integer  $A = A(P) \geq 1$  such that  $P \subset A\Sigma$ . Thus

$$\text{Poly}(nP) \subset \mathcal{P}_{An} \text{ for all } n.$$

Associated to  $P$  we define the *logarithmic indicator function*

$$H_P(z) := \sup_{J \in P} \log |z^J| := \sup_{J \in P} \log [|z_1|^{j_1} \cdots |z_d|^{j_d}].$$

Throughout this paper, we make the assumption on  $P$  that

$$(2.1) \quad \Sigma \subset kP \text{ for some } k \in \mathbb{Z}^+.$$

Under this hypothesis, we have

$$(2.2) \quad H_P(z) \geq \frac{1}{k} \max_{j=1, \dots, d} \log^+ |z_j|.$$

We use  $H_P$  to define generalizations of the Lelong classes  $L(\mathbb{C}^d)$ , the set of all plurisubharmonic (psh) functions  $u$  on  $\mathbb{C}^d$  with the property that  $u(z) - \log |z| = 0(1)$ ,  $|z| \rightarrow \infty$ , and

$$L^+(\mathbb{C}^d) = \{u \in L(\mathbb{C}^d) : u(z) \geq \max_{j=1,\dots,d} \log^+ |z_j| + C_u\}$$

where  $C_u$  is a constant depending on  $u$ . Define

$$L_P = L_P(\mathbb{C}^d) := \{u \in PSH(\mathbb{C}^d) : u(z) - H_P(z) = 0(1), |z| \rightarrow \infty\},$$

and

$$L_{P,+} = L_{P,+}(\mathbb{C}^d) = \{u \in L_P(\mathbb{C}^d) : u(z) \geq H_P(z) + C_u\}.$$

For  $p \in \text{Poly}(nP)$ ,  $n \geq 1$  we have  $\frac{1}{n} \log |p| \in L_P$ ; also each  $u \in L_{P,+}$  is bounded below in  $\mathbb{C}^d$ . We are working on  $\mathbb{C}^d$  instead of  $(\mathbb{C} \setminus 0)^d$  as in [1]. Note  $L_\Sigma = L(\mathbb{C}^d)$  and  $L_{\Sigma,+} = L^+(\mathbb{C}^d)$ .

Given  $E \subset \mathbb{C}^d$ , the  $P$ -extremal function of  $E$  is given by  $V_{P,E}^*(z) := \limsup_{\zeta \rightarrow z} V_{P,E}(\zeta)$  where

$$V_{P,E}(z) := \sup\{u(z) : u \in L_P(\mathbb{C}^d), u \leq 0 \text{ on } E\}.$$

Next, let  $K \subset \mathbb{C}^d$  be closed and let  $w : K \rightarrow \mathbb{R}^+$  be an *admissible weight function on  $K$* :  $w$  is a nonnegative, uppersemicontinuous function with  $\{z \in K : w(z) > 0\}$  nonpluripolar. Letting  $Q := -\log w$ , if  $K$  is unbounded, we additionally require that

$$\liminf_{|z| \rightarrow \infty, z \in K} [Q(z) - H_P(z)] = +\infty.$$

Define the *weighted  $P$ -extremal function*

$$V_{P,K,Q}^*(z) := \limsup_{\zeta \rightarrow z} V_{P,K,Q}(\zeta)$$

where

$$V_{P,K,Q}(z) := \sup\{u(z) : u \in L_P(\mathbb{C}^d), u \leq Q \text{ on } K\}.$$

If  $Q = 0$  we simply write  $V_{P,K,Q} = V_{P,K}$ , consistent with the previous notation. In the case  $P = \Sigma$ ,

$$(2.3) \quad V_{\Sigma,K,Q}(z) = V_{K,Q}(z) := \sup\{u(z) : u \in L(\mathbb{C}^d), u \leq Q \text{ on } K\}$$

is the usual weighed extremal function, e.g., as in Appendix B of [12].

We recall some results in [1], modified for our setting of  $\mathbb{C}^d$  and  $P \subset (\mathbb{R}^+)^d$ . Our hypothesis (2.1) implies Lemma 2.2 in [1] which was used to prove a result on total mixed Monge-Ampère masses and a Siciak-Zaharjuta type theorem. Let  $\omega := dd^c \max_{j=1,\dots,d} \log^+ |z_j|$ .

**Proposition 2.1.** *Let  $P_i \subset (\mathbb{R}^+)^d$ ,  $i = 1, \dots, k$ ,  $k \leq d$ , be convex bodies and let  $u_i, v_i \in L_{P_i} \cap L_{loc}^\infty(\mathbb{C}^d)$ ,  $i = 1, \dots, k$  with*

$$u_i(z) \leq v_i(z) + C_i \text{ for } z \in \mathbb{C}^d, \ i = 1, \dots, k.$$

*Then*

$$\int_{\mathbb{C}^d} dd^c u_1 \wedge \dots \wedge dd^c u_k \wedge \omega^{d-k} \leq \int_{\mathbb{C}^d} dd^c v_1 \wedge \dots \wedge dd^c v_k \wedge \omega^{d-k}.$$

*In particular, if  $u_i \in L_{P_i,+}$ ,  $i = 1, \dots, k$ , then*

$$\int_{\mathbb{C}^d} dd^c u_1 \wedge \dots \wedge dd^c u_k \wedge \omega^{d-k} = M_k$$

*where  $M_k$  is a constant depending only on  $k, d, P_1, \dots, P_k$  (independent of  $u_i \in L_{P_i,+}$ ).*

**Remark 2.2.** The constants  $M_k$  can be computed; see Section 2.1 of [1]. Normalizing so that  $\int_{\mathbb{C}^d} \omega^d = 1$ , for any  $u \in L_{P,+}$  we have

$$(2.4) \quad \int_{\mathbb{C}^d} (dd^c u)^d = \int_{\mathbb{C}^d} (dd^c H_P)^d = d! \text{Vol}(P) =: n_d$$

where  $\text{Vol}(P)$  denotes the euclidean volume of  $P \subset (\mathbb{R}^+)^d$ .

**Proposition 2.3.** *Let  $P \subset (\mathbb{R}^+)^d$  be a convex body,  $K \subset \mathbb{C}^d$  compact, and  $w = e^{-Q}$  an admissible weight on  $K$ . Then*

$$V_{P,K,Q} = \lim_{n \rightarrow \infty} \frac{1}{n} \log \Phi_n = \lim_{n \rightarrow \infty} \frac{1}{n} \log \Phi_{n,P,K,Q}$$

*pointwise on  $\mathbb{C}^d$  where*

$$\Phi_n(z) := \sup\{|p_n(z)| : p_n \in \text{Poly}(nP), \max_{\zeta \in K} |p_n(\zeta)e^{-nQ(\zeta)}| \leq 1\}.$$

*Moreover, if  $Q$  is continuous, i.e.,  $Q \in C(K)$ , and  $V_{P,K,Q}$  is continuous, the convergence is locally uniform on  $\mathbb{C}^d$ .*

**Remark 2.4.** Since  $P \subset A\Sigma$ , we have

$$\Phi_{n,P,K,Q} \leq \Phi_{n,A\Sigma,K,Q}.$$

In particular, for  $Q = 0$ ,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \Phi_{n,P,K,0} = V_{P,K} \leq A \cdot \lim_{n \rightarrow \infty} \frac{1}{An} \log \Phi_{n,A\Sigma,K,0} = A \cdot V_{\Sigma,K}.$$

Thus for any  $K$ ,

$$(2.5) \quad V_{\Sigma,K}^*(z) = 0 \text{ implies } V_{P,K}^*(z) = 0.$$

A compact set  $K \subset \mathbb{C}^d$  is *locally regular* if for all  $z \in K$  and all balls  $B(z, r) := \{w : |w - z| \leq r\}$  we have  $V_{\Sigma, K \cap B(z, r)}^*(z) = 0$ . As examples, the closure of any bounded open set  $D \subset \mathbb{C}^d$  with  $C^1$  boundary is locally regular. It is known (cf., [13], Proposition 2.16) that if  $K$  is locally regular and  $Q \in C(K)$  then  $V_{K, Q}$  in (2.3) is continuous. Using (2.5) for  $K \cap B(z, r)$ , the same proof shows that for *any* convex body  $P \subset (\mathbb{R}^+)^d$ , if  $K$  is locally regular and  $Q \in C(K)$  then  $V_{P, K, Q}$  is continuous.

Following the proofs of Lemma 2.3 and Theorem 2.5 in Appendix B of [12], we have the following.

**Proposition 2.5.** *Let  $P \subset (\mathbb{R}^+)^d$  be a convex body,  $K \subset \mathbb{C}^d$  be closed, and let  $w = e^{-Q}$  be an admissible weight on  $K$ . Then  $S_w := \text{supp}(dd^c V_{P, K, Q}^*)^d$  is compact and*

$$(2.6) \quad \text{supp}((dd^c V_{P, K, Q}^*)^d) \subset \{z \in K : V_{P, K, Q}^*(z) \geq Q(z)\}.$$

Moreover,  $V_{P, K, Q}^* = Q$  q.e. on  $\text{supp}(dd^c V_{P, K, Q}^*)^d$ , i.e., off of a pluripolar set. In particular, if  $Q$  and  $V_{P, K, Q}$  are continuous,

$$\text{supp}((dd^c V_{P, K, Q})^d) \subset \{z \in K : V_{P, K, Q}(z) = Q(z)\}.$$

**Remark 2.6.** It follows under the hypotheses of Proposition 2.5 that

$$V_{P, K, Q}^* = V_{P, S_w, Q|_{S_w}}^* \in L_{P, +}.$$

**Example 2.7.** Let  $P \subset (\mathbb{R}^+)^d$  be a convex body and  $K = T^d$ , the unit  $d$ -torus in  $\mathbb{C}^d$ . Then

$$(2.7) \quad V_{P, T^d}(z) = H_P(z) = \max_{J \in P} \log |z^J| \in L_{P, +}.$$

This is Example 2.3 in [1].

**Remark 2.8.** The results (and proofs) of Propositions 2.1, 2.3 and 2.5, as well as Example 2.7, are valid for  $P \subset (\mathbb{R}^+)^d$  a convex body; some were stated in [1] only in the case of  $P \subset \mathbb{R}^d$  a non-degenerate convex polytope. An alternate proof of (2.7) can be found in [9]. Further explicit examples of weighted  $P$ -extremal functions and their Monge-Ampère measures can be found in [1].

The proof of Theorem 2.6 in Appendix B of [12], which uses a domination principle (Theorem 1.11 in Appendix B of [12]), is valid to obtain the following result.

**Proposition 2.9.** *Let  $P \subset (\mathbb{R}^+)^d$  be a convex body,  $K \subset \mathbb{C}^d$  be closed, and let  $w = e^{-Q}$  be an admissible weight on  $K$ . Then for  $p_n \in \text{Poly}(nP)$  with  $w(z)^n |p_n(z)| \leq M$  q.e.  $z \in S_w$ ,*

$$(2.8) \quad |p_n(z)| \leq M \exp(nV_{P,K,Q}^*(z)), \quad z \in \mathbb{C}^d$$

and

$$w(z)^n |p_n(z)| \leq M \exp[n(V_{P,K,Q}^*(z) - Q(z))], \quad z \in K.$$

Hence  $w(z)^n |p_n(z)| \leq M$  q.e.  $z \in K$ .

For  $K \subset \mathbb{C}^d$  compact,  $w = e^{-Q}$  an admissible weight function on  $K$ , and  $\nu$  a finite measure on  $K$ , we say that the triple  $(K, \nu, Q)$  satisfies a weighted Bernstein-Markov property if for all  $p_n \in \mathcal{P}_n$ ,

$$(2.9) \quad \|w^n p_n\|_K \leq M_n \|w^n p_n\|_{L^2(\nu)} \quad \text{with} \quad \limsup_{n \rightarrow \infty} M_n^{1/n} = 1.$$

Here,  $\|w^n p_n\|_K := \sup_{z \in K} |w(z)^n p_n(z)|$  and

$$(2.10) \quad \|w^n p_n\|_{L^2(\nu)}^2 := \int_K |p_n(z)|^2 w(z)^{2n} d\nu(z).$$

For  $K$  closed but unbounded, we allow  $\nu$  to be locally finite. In this setting, if  $\nu(K) = \infty$  we must assume the weighted  $L^2$ -norms in (2.10) are finite. Next, following [1], given  $P \subset (\mathbb{R}^+)^d$  a convex body, we say that a finite measure  $\nu$  with support in a compact set  $K$  is a Bernstein-Markov measure for the triple  $(P, K, Q)$  if (2.9) holds for all  $p_n \in \text{Poly}(nP)$ . Again for  $K$  closed but unbounded, if  $\nu(K) = \infty$  we must assume the weighted  $L^2$ -norms in (2.10) are finite.

**Remark 2.10.** Since for any  $P$  there exists  $A = A(P) > 0$  with  $\text{Poly}(nP) \subset \mathcal{P}_{An}$  for all  $n$ , if  $(K, \nu, Q)$  satisfies a weighted Bernstein-Markov property, then  $\nu$  is a Bernstein-Markov measure for the triple  $(P, K, \tilde{Q})$  where  $\tilde{Q} = AQ$ . In particular, if  $\nu$  is a *strong Bernstein-Markov measure* for  $K$ ; i.e., if  $\nu$  is a weighted Bernstein-Markov measure for any  $Q \in C(K)$ , then for any such  $Q$ ,  $\nu$  is a Bernstein-Markov measure for the triple  $(P, K, Q)$ .

**Remark 2.11.** In Example 2.7, the monomials  $z^J$ ,  $J \in nP \cap (\mathbb{Z}^+)^d$ , form an orthonormal basis for  $\text{Poly}(nP)$  with respect to normalized Haar measure  $\mu_T$  on  $T^d$ . Moreover,  $\mu_T$  is a strong Bernstein-Markov measure for  $T$  and hence it is a Bernstein-Markov measure for the triple  $(P, T, Q)$  for any  $Q \in C(T)$ .

We refer to [8] for a survey of Bernstein-Markov properties.

**2.2. Projection operator.** To emphasize the relation between the weight  $Q$  and the weighted  $P$ -extremal function  $V_{P,K,Q}^*$ , we may write

$$(2.11) \quad \Pi(Q) = \Pi_K(Q) := V_{P,K,Q}^*.$$

This operator  $\Pi$  is increasing and concave: if  $Q_1 \leq Q_2$  are admissible weights on  $K$ , then  $\Pi(Q_1) \leq \Pi(Q_2)$ ; and if  $0 \leq s \leq 1$  and  $a, a'$  are admissible weights on  $K$ ,

$$(2.12) \quad \Pi(sa + (1-s)a') \geq s\Pi(a) + (1-s)\Pi(a').$$

Since  $sa + (1-s)a'$  is a convex combination of  $a, a'$ , it is an admissible weight on  $K$ . Then (2.12) follows since the right-hand-side is a competitor for the weighted  $P$ -extremal function on the left-hand-side.

It follows from the definition of  $\Pi$ , Proposition 2.5, and Remark 2.6 that  $\Pi$  is Lipschitz on locally regular compacta. That is, if  $a, b \in C(K)$  and  $0 \leq t \leq 1$  then on  $\mathbb{C}^d$ ,

$$(2.13) \quad |\Pi(a + t(b-a)) - \Pi(a)| \leq Ct$$

where  $C = C(a, b) = \max[\sup_{D(0)} |b-a|, \sup_{D(t)} |b-a|]$ . Here  $D(t) := \{\Pi(a + t(b-a)) = a + t(b-a)\}$ . Similarly, if  $u \in C(K)$ , we have, for  $t \in \mathbb{R}$ ,

$$(2.14) \quad |\Pi(a + tu) - \Pi(a)| \leq C|t|$$

where  $C = C(u) = \sup_K |u|$ . In the former case, if  $K$  is unbounded, in order that  $\max[\sup_{D(0)} |b-a|, \sup_{D(t)} |b-a|]$  is a *finite* constant which is independent of  $t$ , we assume that

$$(2.15) \quad \cup_{0 \leq t \leq 1} D(t) \text{ is bounded and } u := b-a \in L^\infty(\cup_{0 \leq t \leq 1} D(t)).$$

Then (2.13) holds. This observation will be used in the proof of Theorem 5.1. In both cases, if  $K$  is compact,  $C$  is finite.

Another result we will need is a comparison principle in  $L_{P,+}$ ; we state and prove the version we will use.

**Proposition 2.12.** *Let  $a_1, a_2 \in L_{P,+}$  and  $b_1, b_2 \in L^+(\mathbb{C}^d)$ . For  $M > 0$ , set  $u_1 := a_1 + Mb_1$  and  $u_2 := a_2 + Mb_2$ . Then*

$$\int_{\{u_1 < u_2\}} (dd^c u_2)^d \leq \int_{\{u_1 < u_2\}} (dd^c u_1)^d.$$

Note that the integrand may be unbounded but each integral is finite by Proposition 2.1.



*Proof.* By adding a constant to  $u_1$ , if necessary, we may assume  $u_1 \geq 0$ . Then for  $\epsilon > 0$ , we have

$$\{(1 + \epsilon)u_1 < u_2\} \subset \{u_1 < u_2\}$$

and  $\{(1 + \epsilon)u_1 < u_2\}$  is bounded. By the standard comparison theorem for locally bounded psh functions on bounded domains (cf., Theorem 3.7.1, [11]),

$$(2.16) \quad \int_{\{(1+\epsilon)u_1 < u_2\}} (dd^c u_2)^d \leq (1 + \epsilon)^d \int_{\{(1+\epsilon)u_1 < u_2\}} (dd^c u_1)^d.$$

Clearly

$$\bigcup_{j=1}^{\infty} \{(1 + 1/j)u_1 < u_2\} = \{u_1 < u_2\}$$

so applying (2.16) with  $\epsilon = 1/j$ , the result follows by monotone convergence upon letting  $j \rightarrow \infty$ .  $\square$

The following lemma (and corollary) will be used in subsection 4.

**Lemma 2.13.** *Let  $a$  be an admissible weight on a compact set  $K$  and let  $u \in C^2(K)$ . Then*

$$(2.17) \quad \lim_{t \rightarrow 0} \int_{D(0) \setminus D(t)} (dd^c \Pi(a))^d = 0$$

where  $D(t) = \{\Pi(a + tu) = a + tu\}$  for  $t \in \mathbb{R}$ .

*Proof.* The hypothesis  $u \in C^2(K)$  means that  $u$  is the restriction to  $K$  of a  $C^2$  function (which we also denote by  $u$ ) on  $\mathbb{C}^d$ ; clearly we can take this function to have compact support. We prove the result for  $t > 0$ ; i.e  $t \rightarrow 0^+$ . We can find  $M > 0$  sufficiently large depending on  $u$  and its support so that  $u + M\psi$  is psh where  $\psi(z) = \frac{1}{2} \log(1 + |z|^2)$ . Observing that

$$D(0) \setminus D(t) \subset S$$

where

$$S := \{\Pi(a + tu) < \Pi(a) + tu\} = \{\Pi(a + tu) + tM\psi < \Pi(a) + t(u + M\psi)\}$$

and

$$D(t) \cap \{\Pi(a + tu) < \Pi(a) + tu\} = \emptyset,$$

we have

$$\int_{D(0) \setminus D(t)} (dd^c \Pi(a))^d \leq \int_S (dd^c \Pi(a))^d$$

$$\begin{aligned}
&\leq \int_S [dd^c(\Pi(a) + t(u + M\psi))]^d \leq \int_S [dd^c(\Pi(a + tu) + tM\psi)]^d \\
&= \int_S [dd^c(\Pi(a + tu))]^d + 0(t) = 0(t).
\end{aligned}$$

Here, the inequality in the second line comes from Proposition 2.12 (with  $M \rightarrow tM$ ).  $\square$

**Corollary 2.14.** *Let  $a, b \in C^2(E)$  be admissible weights on a closed, unbounded set  $E$ . If (2.15) holds then*

$$(2.18) \quad \lim_{t \rightarrow 0} \int_{D(0) \setminus D(t)} (dd^c \Pi(a))^d = 0$$

where  $D(t) = \{\Pi(a + t(b - a)) = a + t(b - a)\}$  for  $0 \leq t \leq 1$ .

*Proof.* First of all,  $(dd^c \Pi(a))^d$  has compact support. Also, by (2.15), the  $P$ -extremal functions  $\Pi(a + t(b - a))$  for all  $0 \leq t \leq 1$  are independent of the values of  $a, b$  outside a large ball. Thus we may assume that  $a = b$  outside a fixed ball. In other words, this case is reduced to the case of Lemma 2.13 where  $u = b - a$ .  $\square$

**Remark 2.15.** For the remainder of this paper,  $K$  will always denote a compact subset of  $\mathbb{C}^d$  while  $E$  will be used for a closed but possibly unbounded subset.

**2.3. Transfinite diameter.** Recall  $d_n$  is the dimension of  $\text{Poly}(nP)$ . We can write

$$\text{Poly}(nP) = \text{span}\{e_1, \dots, e_{d_n}\}$$

where  $\{e_j(z) := z^{\alpha(j)}\}_{j=1, \dots, d_n}$  are the appropriate standard basis monomials. For points  $\zeta_1, \dots, \zeta_{d_n} \in \mathbb{C}^d$ , let

$$\begin{aligned}
(2.19) \quad VDM(\zeta_1, \dots, \zeta_{d_n}) &:= \det[e_i(\zeta_j)]_{i,j=1, \dots, d_n} \\
&= \det \begin{bmatrix} e_1(\zeta_1) & e_1(\zeta_2) & \dots & e_1(\zeta_{d_n}) \\ \vdots & \vdots & \ddots & \vdots \\ e_{d_n}(\zeta_1) & e_{d_n}(\zeta_2) & \dots & e_{d_n}(\zeta_{d_n}) \end{bmatrix}
\end{aligned}$$

and for a compact subset  $K \subset \mathbb{C}^d$  let

$$V_n = V_n(K) := \max_{\zeta_1, \dots, \zeta_{d_n} \in K} |VDM(\zeta_1, \dots, \zeta_{d_n})|.$$

We will show later that the limit

$$(2.20) \quad \delta(K) := \delta(K, P) := \lim_{n \rightarrow \infty} V_n^{1/l_n}$$

exists where  $l_n$  is the sum of the degrees of a set of these basis monomials for  $Poly(nP)$ . We call  $\delta(K)$  the  $P$ -transfinite diameter of  $K$ . More generally, let  $w$  be an admissible weight function on  $K$ . Given  $\zeta_1, \dots, \zeta_{d_n} \in K$ , let

$$\begin{aligned} W(\zeta_1, \dots, \zeta_{d_n}) &:= VDM(\zeta_1, \dots, \zeta_{d_n}) w(\zeta_1)^n \cdots w(\zeta_{d_n})^n \\ &= \det \begin{bmatrix} e_1(\zeta_1) & e_1(\zeta_2) & \cdots & e_1(\zeta_{d_n}) \\ \vdots & \vdots & \ddots & \vdots \\ e_{d_n}(\zeta_1) & e_{d_n}(\zeta_2) & \cdots & e_{d_n}(\zeta_{d_n}) \end{bmatrix} \cdot w(\zeta_1)^n \cdots w(\zeta_{d_n})^n \end{aligned}$$

be a *weighted Vandermonde determinant*. Let

$$(2.21) \quad W_n(K) := \max_{\zeta_1, \dots, \zeta_{d_n} \in K} |W(\zeta_1, \dots, \zeta_{d_n})|$$

and define an  $n$ -th *weighted  $P$ -Fekete set for  $K$  and  $w$*  to be a set of  $d_n$  points  $\zeta_1, \dots, \zeta_{d_n} \in K$  with the property that

$$|W(\zeta_1, \dots, \zeta_{d_n})| = W_n(K).$$

We also write  $\delta^{w,n}(K) := W_n(K)^{1/l_n}$  and we will show, more generally, that the *weighted  $P$ -transfinite diameter*

$$(2.22) \quad \delta^w(K) := \delta^w(K, P) := \lim_{n \rightarrow \infty} \delta^{w,n}(K) := \lim_{n \rightarrow \infty} W_n(K)^{1/l_n}$$

exists. For each  $n$ , if we take points  $z_1^{(n)}, z_2^{(n)}, \dots, z_{d_n}^{(n)} \in K$  for which

$$(2.23) \quad \lim_{n \rightarrow \infty} [|VDM(z_1^{(n)}, \dots, z_{d_n}^{(n)})| w(z_1^{(n)})^n w(z_2^{(n)})^n \cdots w(z_{d_n}^{(n)})^n]^{\frac{1}{l_n}} = \delta^w(K)$$

– we call these *asymptotically weighted  $P$ -Fekete arrays* – and we let  $\mu_n := \frac{1}{d_n} \sum_{j=1}^{d_n} \delta_{z_j^{(n)}}$ , one of our results, Corollary 6.5, is that

$$\mu_n \rightarrow \frac{1}{n_d} (dd^c \Pi(Q))^d \text{ weak} - *.$$

(recall (2.4)).

**Remark 2.16.** For  $P = \Sigma$  so that  $Poly(n\Sigma) = \mathcal{P}_n$ , we have

$$d_n(\Sigma) = \binom{d+n}{d} = 0(n^d/d!) \text{ and } l_n(\Sigma) = \frac{d}{d+1} n d_n(\Sigma)$$

In particular,

$$\frac{l_n(\Sigma)}{d_n(\Sigma)} = \frac{nd}{d+1}.$$

For a general convex body  $P \subset (\mathbb{R}^+)^d$  with  $A > 0$  so that  $P \subset A\Sigma$ , we write

$$(2.24) \quad l_n = f_n(d) \frac{nd}{d+1} d_n = f_n(d) \frac{l_n(\Sigma)}{d_n(\Sigma)} d_n.$$

We will need to know that  $l_n/d_n$  divided by  $l_n(\Sigma)/d_n(\Sigma)$  has a limit; i.e., that

$$(2.25) \quad \lim_{n \rightarrow \infty} f_n(d) =: \mathcal{A} = \mathcal{A}(P, d)$$

exists. It suffices to verify (2.25) for  $P \subset (\mathbb{R}^+)^d$  a non-degenerate convex polytope. It follows from Theorem 2 of Lecture 2 in [14]

- (1) applied to  $f(j_1, \dots, j_d) \equiv 1$  that  $d_n$  is a polynomial of degree  $d$  in  $n$  with

$$d_n = \text{Vol}(P)n^d + 0(n^{d-1}); \text{ and}$$

- (2) applied to  $f(j_1, \dots, j_d) = j_1 + \dots + j_d$  that  $l_n$  is a polynomial of degree  $d+1$  in  $n$  with

$$l_n = C_P n^{d+1} + 0(n^d)$$

$$\text{where } C_P = \int_P (x_1 + \dots + x_d) dx_1 \dots dx_d.$$

Thus

$$l_n/d_n = \frac{C_P n^{d+1} + 0(n^d)}{\text{Vol}(P)n^d + 0(n^{d-1})} = \frac{nC_P}{\text{Vol}(P)} + 0(1)$$

which proves (2.25):

$$f_n(d) = \frac{(d+1)l_n}{ndd_n} = \frac{(d+1)}{d} \frac{l_n}{nd_n} \rightarrow \frac{(d+1)}{d} \frac{C_P}{\text{Vol}(P)}.$$

**2.4. Gram matrices and  $P$ -optimal measures.** Let  $E \subset \mathbb{C}^d$  be closed and let  $w$  be an admissible weight on  $E$ . We take  $\mu$  a locally finite measure on  $E$  and for each  $n$  we define a weighted inner product on  $\text{Poly}(nP)$ :

$$(2.26) \quad \langle f, g \rangle_{\mu, w} := \int_E f(z) \overline{g(z)} w(z)^{2n} d\mu.$$

We assume that  $\|f\|_{L^2(w^n d\mu)}^2 = \langle f, f \rangle_{\mu, w} < \infty$  for all  $f \in \text{Poly}(nP)$  and that (2.26) is non-degenerate in the sense that  $\|f\|_{L^2(w^n d\mu)} = 0$  implies  $f \equiv 0$ . Fixing a basis  $\beta_n = \{p_1, p_2, \dots, p_{d_n}\}$  of  $\text{Poly}(nP)$  we form the Gram matrix

$$G_n^{\mu, w} = G_n^{\mu, w}(\beta_n) := [\langle p_i, p_j \rangle_{\mu, w}] \in \mathbb{C}^{d_n \times d_n}$$

and the associated  $n$ -th Bergman function

$$(2.27) \quad B_n^{\mu,w}(z) := \sum_{j=1}^{d_n} |q_j(z)|^2 w(z)^{2n}$$

where  $Q_n = \{q_1, q_2, \dots, q_{d_n}\}$  is an orthonormal basis for  $Poly(nP)$  with respect to the inner-product (2.26). We make an observation which will be used in Lemma 2.17 below. With this basis  $\beta_n$ , if we write

$$(2.28) \quad P(z) = \begin{bmatrix} p_1(z) \\ p_2(z) \\ \vdots \\ p_{d_n}(z) \end{bmatrix} \in \mathbb{C}^{d_n}$$

then

$$(2.29) \quad w(z)^{2n} P(z)^* (G_n^{\mu,w}(\beta_n))^{-1} P(z) = B_n^{\mu,w}(z).$$

To see this,  $G := G_n^{\mu,w}(\beta_n)$  and  $G^{-1}$  are positive definite, Hermitian matrices; hence  $G^{1/2}$ ,  $G^{-1/2} := (G^{-1})^{1/2}$  exist; writing  $P := P(z)$ , we have

$$P^* G^{-1} P = P^* G^{-1/2} G^{-1/2} P = (G^{-1/2} P)^* G^{-1/2} P.$$

To verify that  $w(z)^{2n}$  times the right-hand-side yields  $B_n^{\mu,w}(z)$ , note that since  $G = \int_E P P^* w^{2n} d\mu$ , the polynomials  $\{\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_{d_n}\}$  defined by

$$(2.30) \quad G^{-1/2} P := \begin{bmatrix} \tilde{p}_1(z) \\ \tilde{p}_2(z) \\ \vdots \\ \tilde{p}_{d_n}(z) \end{bmatrix} \in \mathbb{C}^{d_n}$$

form an *orthonormal* basis for  $Poly(nP)$  in  $L^2(\mu)$ : for

$$\begin{aligned} \int_E G^{-1/2} P \cdot (G^{-1/2} P)^* w^{2n} d\mu &= G^{-1/2} \left[ \int_E P P^* w^{2n} d\mu \right] G^{1/2} \\ &= G^{-1/2} G G^{1/2} = I, \end{aligned}$$

the  $d_n \times d_n$  identity matrix. Thus

$$B_n^{\mu,w}(z) = \sum_{j=1}^{d_n} |\tilde{p}_j(z)|^2 w(z)^{2n} = w^{2n} (G^{-1/2} P)^* G^{-1/2} P.$$

Given  $E$ , and  $w$  on  $E$ , for a function  $u \in C(E)$ , we consider the weight  $w_t(z) := w(z) \exp(-tu(z))$ ,  $t \in \mathbb{R}$ . Apriori,  $w_t$  need not be admissible. Let  $\{\mu_n\}$  be a sequence of measures on  $E$ . Fixing a basis  $\beta_n := \{p_1, \dots, p_{d_n}\}$  of  $\text{Poly}(nP)$ , we set

$$(2.31) \quad f_n(t) := -\frac{1}{2l_n} \log \det(G_n^{\mu_n, w_t})$$

where  $G_n^{\mu_n, w_t} = G_n^{\mu_n, w_t}(\beta_n)$ . We have the following result (Lemma 5.1 in [4] or Lemma 3.5 in [7]) which will be used to prove Theorem 5.1.

**Lemma 2.17.** *Suppose  $w_t$  is admissible for  $t$  in an interval containing 0. For such  $t$ , we have*

$$f'_n(t) = \frac{n}{l_n} \int_E u(z) B_n^{\mu_n, w_t}(z) d\mu_n.$$

*Proof.* Recall that  $G_n^{\mu_n, w_t}$  is a positive definite Hermitian matrix; hence we can define  $\log(G_n^{\mu_n, w_t})$ . Using  $\log \det(G_n^{\mu_n, w_t}) = \text{trace} \log(G_n^{\mu_n, w_t})$ , we calculate

$$\begin{aligned} 2l_n f'_n(t) &= -\frac{d}{dt} \text{trace}(\log(G_n^{\mu_n, w_t})) \\ &= -\text{trace} \left( \frac{d}{dt} \log(G_n^{\mu_n, w_t}) \right) \\ &= -\text{trace} \left( (G_n^{\mu_n, w_t})^{-1} \frac{d}{dt} G_n^{\mu_n, w_t} \right) \\ &= 2n \text{trace} \left( (G_n^{\mu_n, w_t})^{-1} \left[ \int_E p_i(z) \overline{p_j(z)} u(z) w(z)^{2n} \exp(-2ntu(z)) d\mu_n \right] \right). \end{aligned}$$

We use

$$\text{trace}(ABC) = \text{trace}(CAB) = \text{trace}(BCA)$$

to write the previous line as

$$\begin{aligned} &= 2n \int_E P^*(z) (G_n^{\mu_n, w_t})^{-1} P(z) u(z) w(z)^{2n} \exp(-2ntu(z)) d\mu_n \\ &= 2n \int_E u(z) P^*(z) (G_n^{\mu_n, w_t})^{-1} P(z) w_t(z)^{2n} d\mu_n \\ &= 2n \int_E u(z) B_n^{\mu_n, w_t}(z) d\mu_n \end{aligned}$$

where the last equality follows from (2.29):

$$w_t^{2n} P^* (G_n^{\mu_n, w_t})^{-1} P = B_n^{\mu_n, w_t}.$$

□

Similar, but more involved calculations, give the following (cf., Lemma 3.6 of [7]).

**Lemma 2.18.** *The functions  $f_n(t)$  are concave, i.e.,  $f_n''(t) \leq 0$ .*

Now we restrict to  $K \subset \mathbb{C}^d$  compact and non-pluripolar. Fix  $\mu$  a probability measure on  $K$  and  $w$  an admissible weight on  $K$ . If  $\mu$  has the property that

$$(2.32) \quad \det(G_n^{\mu',w}) \leq \det(G_n^{\mu,w})$$

for all other probability measures  $\mu'$  on  $K$  then  $\mu$  is said to be a *P-optimal measure of degree  $n$  for  $K$  and  $w$* . This property is independent of the basis used for  $Poly(nP)$ . An equivalent characterization is that

$$\max_{z \in K} B_n^{\mu,w}(z) \leq \max_{z \in K} B_n^{\mu',w}(z)$$

for all other probability measures  $\mu'$  on  $K$ . Note that for *any* probability measure  $\mu'$ ,  $\int_K B_n^{\mu',w}(z) d\mu' = d_n$ , so that

$$\max_{z \in K} B_n^{\mu',w}(z) \geq d_n.$$

For a *P*-optimal measure we have equality.

**Proposition 2.19.** *Let  $w$  be an admissible weight on  $K$ . A probability measure  $\mu$  is a *P-optimal measure of degree  $n$  for  $K$  and  $w$*  if and only if*

$$\max_{z \in K} B_n^{\mu,w}(z) = d_n.$$

It follows that if  $\mu$  is *P*-optimal for  $K$  and  $w$  then

$$(2.33) \quad B_n^{\mu,w}(z) = d_n, \quad a.e. \mu.$$

We omit the proof; cf., [10] or Proposition 3.1 of [7].

**2.5. Ball volume ratios.** Given a (complex)  $M$ -dimensional vector space  $V$ , and two subsets  $A, B$  in  $V$ , we write

$$[A : B] := \log \frac{\text{vol}(A)}{\text{vol}(B)}$$

where “vol” denotes any (Haar) measure on  $V$  (taking the ratio makes  $[A : B]$  independent of this choice). In particular, if  $V$  is equipped with

two Hermitian inner products  $h, h'$ , and  $B, B'$  are the corresponding unit balls, then a linear algebra exercise shows that

$$(2.34) \quad [B : B'] = \log \det[h'(e_i, e_j)]_{i,j=1,\dots,M}$$

where  $e_1, \dots, e_M$  is an  $h$ -orthonormal basis for  $V$ . In other words,  $[B : B']$  is a *Gram determinant* with respect to the  $h'$  inner product relative to the  $h$ -orthonormal basis. Indeed,  $[B : B']$  is independent of the  $h$ -orthonormal basis chosen for  $V$ .

We will generally take  $V = \text{Poly}(nP)$  and our subsets to be unit balls with respect to norms on  $\text{Poly}(nP)$ ; in this case we call (2.34) a *ball volume ratio*. In particular, given  $P$ , let  $\mu$  be a locally finite measure on a closed set  $E \subset \mathbb{C}^d$ , and let  $w$  be an admissible weight on  $E$  such that (2.26) is non-degenerate and  $\|f\|_{L^2(w^n d\mu)}^2 < \infty$  for all  $f \in \text{Poly}(nP)$ . We noted that for the unit torus  $T^d$ , the standard basis monomials  $\beta_n = \{z^J, J \in nP \cap (\mathbb{Z}^+)^d\}$  form an orthonormal basis for  $\text{Poly}(nP)$  with respect to the standard Haar measure  $\mu_T$  on  $T^d$ . Letting

$$B_n = \{p_n \in \text{Poly}(nP) : \|p_n w^n\|_{L^2(\mu)} = \|p_n\|_{L^2(w^{2n}\mu)} \leq 1\}$$

and

$$B'_n = \{p_n \in \text{Poly}(nP) : \|p_n\|_{L^2(\mu_T)} \leq 1\}$$

be  $L^2$ -balls in  $\text{Poly}(nP)$ , we have

$$(2.35) \quad [B_n : B'_n] = \log \det G_n^{\mu,w}(\beta_n).$$

We will also use  $L^\infty$ -balls in  $\text{Poly}(nP)$ .

Taking  $E = K$  compact and  $\mu$  finite, replacing the standard basis monomials  $\{z^J, J \in nP \cap (\mathbb{Z}^+)^d\}$  by orthogonal polynomials  $\{r_J(z)\}$  using the Gram-Schmidt process in  $L^2(w^{2n}\mu)$ , the Gram determinants  $\det(G_n^{\mu,w}) = \prod_J \|r_J\|_{L^2(w^{2n}\mu)}^2$  are unchanged and we have

$$\det(G_n^{\mu,w}) = \frac{1}{d_n!} Z_n := \frac{1}{d_n!} Z_n(\mu, w)$$

where

$$Z_n := \int_{K^{d_n}} |VDM(z_1, \dots, z_{d_n})|^2 w(z_1)^{2n} \cdots w(z_{d_n})^{2n} d\mu(z_1) \cdots d\mu(z_{d_n}).$$

It is easy to see that if  $\mu$  is a Bernstein-Markov measure for the triple  $(P, K, Q)$  where  $w = e^{-Q}$ , i.e., (2.9) holds for  $\mu$ , then

$$(2.36) \quad Z_n \leq \delta^{w,n}(K)^{2l_n} \mu(K)^{d_n} \leq \mu(K)^{d_n} M_n^{2d_n} Z_n.$$



**Conjecture 2.20.** Let  $K \subset \mathbb{C}^d$  be compact and let  $w = e^{-Q}$  be an admissible weight on  $K$ . If  $\mu$  is a Bernstein-Markov measure for the triple  $(P, K, Q)$ , then

$$(2.37) \quad \lim_{n \rightarrow \infty} Z_n^{\frac{1}{2l_n}} = \lim_{n \rightarrow \infty} \det(G_n^{\mu, w})^{\frac{1}{2l_n}} =: \mathcal{F}_P(K, Q)$$

exists.

We verify the conjecture in Remark 5.2. It then follows from (2.36) and (2.25) that  $\lim_{n \rightarrow \infty} \delta^{w, n}(K)$  exists and equals  $\mathcal{F}_P(K, Q)$ . This gives the existence of the limit in the definition of the  $P$ –transfinite diameter (2.20) and the weighted  $P$ –transfinite diameter (2.22). We also have:

**Proposition 2.21.** *Let  $K$  be compact and  $w$  an admissible weight function. Assume (2.37). For  $n = 1, 2, \dots$ , let  $\mu_n$  be a  $P$ –optimal measure of order  $n$  for  $K$  and  $w$ . Then*

$$\lim_{n \rightarrow \infty} \det(G_n^{\mu_n, w})^{\frac{1}{2l_n}} = \mathcal{F}_P(K, Q).$$

*Proof.* We will use

$$\begin{aligned} \int_{K^{d_n}} |VDM(z_1, \dots, z_{d_n})|^2 w(z_1)^{2n} \cdots w(z_{d_n})^{2n} d\mu_n(z_1) \cdots d\mu_n(z_{d_n}) \\ = d_n! \det(G_n^{\mu_n, w}). \end{aligned}$$

It follows, since  $\mu_n$  is a probability measure, that

$$\det(G_n^{\mu_n, w}) \leq \frac{1}{d_n!} (\delta_n^w(K))^{2l_n}.$$

Now if  $f_1, f_2, \dots, f_{d_n} \in K$  are weighted  $P$ –Fekete points of order  $n$  for  $K$ , i.e., points in  $K$  for which

$$|VDM(z_1, \dots, z_{d_n})| w^n(z_1) \cdots w^n(z_{d_n})$$

is maximal, then the discrete measure

$$(2.38) \quad \nu_n = \frac{1}{d_n} \sum_{k=1}^{d_n} \delta_{f_k}$$

is a candidate for a  $P$ –optimal measure of order  $n$ ; hence

$$\det(G_n^{\nu_n, w}) \leq \det(G_n^{\mu_n, w}).$$

But

$$\det(G_n^{\nu_n, w}) = \frac{1}{d_n^{d_n}} |VDM(f_1, \dots, f_{d_n})|^2 w(f_1)^{2n} \cdots w(f_{d_n})^{2n}$$

$$= \frac{1}{d_n^{d_n}} (\delta^{w,n}(K))^{2l_n}$$

so that

$$\frac{1}{d_n^{d_n}} (\delta^{w,n}(K))^{2l_n} \leq \det(G_n^{\mu_n, w}).$$

The result follows since (2.37) implies  $\lim_{n \rightarrow \infty} \delta^{w,n}(K)$  exists and equals  $\mathcal{F}_P(K, Q)$ .  $\square$

For future use we note that the ball volume ratios satisfy  $[A : B] = -[B : A]$ ; the cocycle condition:

$$[A : B] + [B : C] + [C : A] = 0;$$

and they are “monotone” in the first slot: for any  $B \subset \text{Poly}(nP)$ , if  $E \subset \mathbb{C}^d$  is closed with admissible weights  $Q_1 \leq Q_2$  and

$$\mathcal{B}^\infty(E, nQ_i) := \{p_n \in \text{Poly}(nP) : \|p_n e^{-nQ_i}\|_E \leq 1\}, \quad i = 1, 2$$

then

$$(2.39) \quad [\mathcal{B}^\infty(E, nQ_1) : B] \leq [\mathcal{B}^\infty(E, nQ_2) : B]$$

(with a similar statement for  $L^2$ -balls for  $\mu$  a measure on  $E$ ). Analogous properties will hold for the energy functional discussed next.

### 3. ENERGY.

For  $u, v \in L_{P,+}$ , we define the *energy*

$$(3.1) \quad \mathcal{E}(u, v) := \int_{\mathbb{C}^d} (u - v) \sum_{j=0}^d (dd^c u)^j \wedge (dd^c v)^{d-j}.$$

A reason for this definition will appear in Proposition 3.1, and Theorem 5.1 will relate asymptotics of certain ball volume ratios to the energy of appropriate  $u, v$ . For any functions  $A, B \in L_{P,+}$  we have  $A - B$  is uniformly bounded on  $\mathbb{C}^d$ . We will need an integration by parts formula in this setting. Using results from Bedford-Taylor [2], one can show: given  $A, B, C, D \in L_{P,+}$ , let  $u_1, \dots, u_{d-1} \in L_{P,+}$ . Then

$$\begin{aligned} (3.2) \quad & \int_{\mathbb{C}^d} (A - B)(dd^c C - dd^c D) \wedge dd^c u_1 \wedge \dots \wedge dd^c u_{d-1} \\ &= \int_{\mathbb{C}^d} (C - D)(dd^c A - dd^c B) \wedge dd^c u_1 \wedge \dots \wedge dd^c u_{d-1} \\ &= - \int_{\mathbb{C}^d} d(A - B) \wedge d^c(C - D) \wedge dd^c u_1 \wedge \dots \wedge dd^c u_{d-1}. \end{aligned}$$

The proof of the following fundamental differentiability property of the energy is exactly as that of Proposition 4.1 of [4].

**Proposition 3.1.** *Let  $u, u', v \in L_{P,+}$ . For  $0 \leq t \leq 1$ , let*

$$f(t) := \mathcal{E}(u + t(u' - u), v).$$

*Then  $f'(t)$  exists for  $0 \leq t \leq 1$  and*

$$(3.3) \quad f'(t) = (d+1) \int_{\mathbb{C}^d} (u' - u)(dd^c(u + t(u' - u)))^d.$$

**Remark 3.2.** Here we mean the appropriate one-sided derivatives at  $t = 0$  and  $t = 1$ ; e.g.,

$$(3.4) \quad f'(0) := \lim_{t \rightarrow 0^+} \frac{f(t) - f(0)}{t} = (d+1) \int_{\mathbb{C}^d} (u' - u)(dd^c u)^d.$$

This last statement implies (3.3). For if  $s$  is fixed,

$$g(t) := f(s+t) = \mathcal{E}(u + (s+t)(u' - u), v) = \mathcal{E}(u + s(u' - u) + t(u' - u), v)$$

and applying (3.4) to  $g$  (so  $u \rightarrow u + s(u' - u)$ ) we get

$$g'(0) = f'(s) = (d+1) \int_{\mathbb{C}^d} (u' - u)(dd^c(u + s(u' - u)))^d.$$

We sometimes write (3.4) in “directional derivative” notation as

$$(3.5) \quad \langle \mathcal{E}'(u), u' - u \rangle = (d+1) \int_{\mathbb{C}^d} (u' - u)(dd^c u)^d.$$

Note that the differentiation formula (3.3) is independent of  $v$ . This also follows from the *cocycle property*:

**Proposition 3.3.** *Let  $u, v, w \in L_{P,+}$ . Then*

$$\mathcal{E}(u, v) + \mathcal{E}(v, w) + \mathcal{E}(w, u) = 0.$$

*Proof.* Let

$$f(t) := \mathcal{E}(u + t(w - u), v) + \mathcal{E}(v, u)$$

and

$$g(t) := \mathcal{E}(u + t(w - u), w) + \mathcal{E}(w, u).$$

Then  $f(0) = g(0) = 0$  by antisymmetry of  $\mathcal{E}$ . From (3.3),

$$f'(t) = (d+1) \int_{\mathbb{C}^d} (w - u)(dd^c(u + t(w - u)))^d = g'(t)$$

for all  $t$ . Thus  $f(1) = g(1)$ ; i.e.,

$$\mathcal{E}(w, v) + \mathcal{E}(v, u) = \mathcal{E}(w, w) + \mathcal{E}(w, u) = \mathcal{E}(w, u).$$

□

The independence of (3.3) on  $v$  now follows: if  $v, v' \in L_{P,+}$ , then

$$\mathcal{E}(u + t(u' - u), v') + \mathcal{E}(v', v) + \mathcal{E}(v, u + t(u' - u)) = 0$$

so that the difference

$$\mathcal{E}(u + t(u' - u), v') - \mathcal{E}(u + t(u' - u), v) = \mathcal{E}(v, v')$$

is independent of  $t$ . Thus we consider  $\mathcal{E}$  as a functional on the first slot with the second fixed. As such, it is *increasing and concave*; the proof is exactly as for Proposition 4.4 of [4] and requires formula (3.2).

**Proposition 3.4.** *Let  $u, v, w \in L_{P,+}$ . Then*

$$u \geq v \text{ implies } \mathcal{E}(u, w) \geq \mathcal{E}(v, w)$$

and for  $0 \leq t \leq 1$

$$\mathcal{E}(tu + (1-t)v, w) \geq t\mathcal{E}(u, w) + (1-t)\mathcal{E}(v, w);$$

i.e.,  $g(t) := \mathcal{E}(tu + (1-t)v, w)$  satisfies  $g''(t) \leq 0$ .

A consequence of concavity is the following. Let  $u_1, u_2, v \in L_{P,+}$ . Letting

$$g(s) := \mathcal{E}(u_1 + s(u_2 - u_1), v)$$

for  $0 \leq s \leq 1$ , we have concavity of  $g$  so that  $g(s) \leq g(0) + g'(0)s$ . In particular, at  $s = 1$ , we have  $g(1) \leq g(0) + g'(0)$ ; i.e.,

$$(3.6) \quad \mathcal{E}(u_2, v) \leq \mathcal{E}(u_1, v) + (d+1) \int_{\mathbb{C}^d} (u_2 - u_1)(dd^c u_1)^d.$$

For future use, we record the following.

**Lemma 3.5.** *Let  $\{w_j\}, \{v_j\} \subset L_{P,+}$  with  $w_j \uparrow w \in L_{P,+}$  and  $v_j \uparrow v \in L_{P,+}$ . Then*

$$\mathcal{E}(w_j, v) \rightarrow \mathcal{E}(w, v) \text{ and } \mathcal{E}(w_j, v_j) \rightarrow \mathcal{E}(w, v).$$

*Proof.* From Proposition 3.3, it suffices to prove the first statement. This follows directly from the *proof* of Lemma 6.3 of [2]: *given*

$$w, \{v_j\}, v, \{u_{1,j}\}, u_1, \dots, \{u_{d,j}\}, u_d \text{ in } L_{P,+}$$

with  $v_j \uparrow v$ ,  $u_{1,j} \uparrow u_1, \dots, u_{d,j} \uparrow u_d$ ,

$$\lim_{j \rightarrow \infty} \int_{\mathbb{C}^d} (w - v_j) dd^c u_{1,j} \wedge \dots \wedge dd^c u_{d,j} = \int_{\mathbb{C}^d} (w - v) dd^c u_1 \wedge \dots \wedge dd^c u_d.$$

□

We remark that if  $w_j \downarrow w \in L_{P,+}$  and  $v_j \downarrow v \in L_{P,+}$  then we still have

$$(3.7) \quad \mathcal{E}(w_j, v) \rightarrow \mathcal{E}(w, v) \text{ and } \mathcal{E}(w_j, v_j) \rightarrow \mathcal{E}(w, v).$$

The first statement is standard and the second follows from the first by Proposition 3.3.

#### 4. DIFFERENTIABILITY OF $\mathcal{E} \circ \Pi$ .

We turn to the main differentiability result. Our exposition mimics Lemmas 4.10 and 4.11 of [4]; since this is the key ingredient in proving Theorem 5.1 we include all details. Generally we will fix a function  $v \in L_{P,+}$  which will be in the second slot of all energy terms and we simply write, for any  $\tilde{v} \in L_{P,+}$ ,

$$\mathcal{E}(\tilde{v}) := \mathcal{E}(\tilde{v}, v).$$

If we need to emphasize a specific  $v$ , we revert to the notation on the right-hand-side of this equation. Recall for  $E \subset \mathbb{C}^d$  closed and an admissible weight  $a$  on  $E$ , we write  $\Pi(a)$  (sometimes  $\Pi_E(a)$ ) to denote the regularized weighted  $P$ -extremal function  $V_{P,E,a}^*$ .

We state two versions of differentiability of  $\mathcal{E} \circ \Pi$ . One version, Proposition 4.1, is for a second admissible weight  $b$  on  $E$  where we consider the perturbed weight  $a + t(b - a)$  and the associated weighted  $P$ -extremal function  $\Pi(a + t(b - a))$  and we show the differentiability of

$$F(t) := \mathcal{E}(\Pi(a + t(b - a))).$$

Taking  $v = \Pi(a)$ , as we will in Propositions 4.1, 4.2 and Lemma 4.3,

$$(4.1) \quad F(0) = \mathcal{E}(\Pi(a)) = \mathcal{E}(\Pi(a), \Pi(a)) = 0.$$

If  $E$  is unbounded, we will need to make an additional assumption on  $u := b - a$  so that (2.13) holds; also, in this case, we restrict to  $0 \leq t \leq 1$  so that  $a + t(b - a) = tb + (1 - t)a$ , being a convex combination of  $a, b$ , is admissible on  $E$ . The second version of differentiability for  $\mathcal{E} \circ \Pi$ , Proposition 4.2, is for a compact set  $K$  and an arbitrary real  $t$ . We

take a function  $u \in C(K)$ , consider the perturbed weight  $a + tu$ , and show the differentiability of

$$F(t) := \mathcal{E}(\Pi(a + tu)).$$

Apriori, since  $t \in \mathbb{R}$ , we must assume  $u$  is continuous so that  $a + tu$  is an admissible (lowersemicontinuous) weight. The following results utilize Lemma 2.13 and Corollary 2.14; hence we assume  $C^2$ -regularity of  $a, b$  and/or  $u$ .

**Proposition 4.1.** *Let  $v \in L_{P,+}$ . For admissible weights  $a, b \in C^2(E)$  on a closed set  $E \subset \mathbb{C}^d$ , let  $u := b - a$  and let*

$$F(t) := \mathcal{E}(\Pi(a + tu), v)$$

*for  $t \in \mathbb{R}$ . If  $E$  is unbounded, assume (2.15) holds and  $0 \leq t \leq 1$ . Then*

$$(4.2) \quad F'(t) = (d+1) \int_{\mathbb{C}^d} u(dd^c \Pi(a + tu))^d.$$

**Proposition 4.2.** *Let  $v \in L_{P,+}$ . For an admissible weight  $a$  on a compact set  $K \subset \mathbb{C}^d$  and  $u \in C^2(K)$ , let*

$$F(t) := \mathcal{E}(\Pi(a + tu), v)$$

*for  $t \in \mathbb{R}$ . Then*

$$(4.3) \quad F'(t) = (d+1) \int_{\mathbb{C}^d} u(dd^c \Pi(a + tu))^d.$$

We prove Propositions 4.1 and 4.2 simultaneously.

*Proof.* We may take  $v = \Pi(a)$ . As in the proof of Proposition 3.1 we prove only the one-sided limit as  $t \rightarrow 0^+$ :

$$(4.4) \quad F'(0) := \lim_{t \rightarrow 0^+} \frac{F(t) - F(0)}{t} = (d+1) \int_{\mathbb{C}^d} u(dd^c \Pi(a))^d.$$

This implies (4.2). For if  $s$  is fixed,

$$\begin{aligned} G(t) &:= F(s+t) = \mathcal{E}(\Pi(a + (s+t)u), v) \\ &= \mathcal{E}(\Pi(a + su + tu), v) \end{aligned}$$

and applying (4.4) to  $G$  (so  $a \rightarrow a + su$ ) we get

$$G'(0) = F'(s) = (d+1) \int_{\mathbb{C}^d} u(dd^c \Pi(a + su))^d.$$

Note that  $F(0) = 0$  (see (4.1)) and to verify (4.4) it suffices to prove

$$(4.5) \quad \mathcal{E}(\Pi(a+tu), \Pi(a)) = (d+1)t \int_{\mathbb{C}^d} u(dd^c \Pi(a))^d + o(t).$$

We need two ingredients for (4.5):

$$(4.6) \quad \mathcal{E}(\Pi(a+tu), \Pi(a)) = (d+1) \int_{\mathbb{C}^d} [\Pi(a+tu) - \Pi(a)](dd^c \Pi(a))^d + o(t)$$

and

$$(4.7) \quad \lim_{t \rightarrow 0} \int_{D(0) \setminus D(t)} (dd^c \Pi(a))^d = 0$$

where

$$D(t) := \{z \in \mathbb{C}^d : \Pi(a+tu)(z) = (a+tu)(z)\}.$$

We have proved (4.7) in Lemma 2.13.

We state and prove (4.6) in a separate lemma. Given (4.6) and (4.7), and observing from (2.6) that

$$(4.8) \quad \text{supp}(dd^c \Pi(a))^d \subset D(0),$$

(4.5) follows as in [4], p. 28:

$$\begin{aligned} \mathcal{E}(\Pi(a+tu), \Pi(a)) &= (d+1) \int_{\mathbb{C}^d} [\Pi(a+tu) - \Pi(a)](dd^c \Pi(a))^d + o(t) \\ &= (d+1) \int_{D(0) \setminus D(t)} [\Pi(a+tu) - \Pi(a)](dd^c \Pi(a))^d \\ &\quad + (d+1) \int_{D(0) \cap D(t)} [\Pi(a+tu) - \Pi(a)](dd^c \Pi(a))^d + o(t) \\ &= (d+1) \int_{D(0) \setminus D(t)} [\Pi(a+tu) - \Pi(a)](dd^c \Pi(a))^d \\ &\quad + (d+1)t \int_{D(0) \cap D(t)} u(dd^c \Pi(a))^d + o(t) \\ &= (d+1) \int_{D(0) \setminus D(t)} [\Pi(a+tu) - \Pi(a) - tu](dd^c \Pi(a))^d \\ &\quad + (d+1)t \int_{D(0)} u(dd^c \Pi(a))^d + o(t) \end{aligned}$$

since  $\Pi(a+tu) - \Pi(a) = tu$  on  $D(0) \cap D(t)$ . Now (2.13) or (2.14) implies

$$|\Pi(a+tu) - \Pi(a) - tu| = 0(t)$$

on the *bounded* set  $D(0) \setminus D(t)$  (recall if  $E$  is unbounded we assume (2.15) holds in the setting of Proposition 4.1) and this fact, combined with (4.7) and (4.8), finishes the proof.  $\square$

In (4.6), since  $(dd^c\Pi(a))^d$  is supported in  $D(0)$ ,

$$\int_{\mathbb{C}^d} [\Pi(a + tu) - \Pi(a)](dd^c\Pi(a))^d = \int_{D(0)} [\Pi(a + tu) - \Pi(a)](dd^c\Pi(a))^d;$$

and, on  $D(t) \cap D(0)$ , we have  $\Pi(a + tu) - \Pi(a) = tu$ . The content of (4.7) is that the contribution to this integral on  $D(0) \setminus D(t)$  is negligible. The content of (4.6), Lemma 4.3 below, is that the contribution of each of the  $d + 1$  terms in the energy  $\mathcal{E}(\Pi(a + tu), \Pi(a))$  is the same, up to  $o(t)$ , as that involving the term  $(dd^c\Pi(a))^d$ . Again we write

$$\begin{aligned} F(t) &:= \mathcal{E}(\Pi(a + tu)) = \mathcal{E}(\Pi(a + tu), \Pi(a)) \\ &= \int [\Pi(a + tu) - \Pi(a)][(dd^c\Pi(a + tu))^d + \dots + (dd^c\Pi(a))^d]. \end{aligned}$$

Another interpretation of (4.6) is that to prove the differentiability of  $\mathcal{E} \circ \Pi$ , we can replace  $\mathcal{E}$  by its “linearization” at  $\Pi(a)$ . As in previous arguments, we only give the proof at  $t = 0$  and for the one-sided limit in (4.3) as  $t \rightarrow 0^+$ . The next result does not require smoothness of  $u$ .

**Lemma 4.3.** *For an admissible weight  $a$  on  $E$  and  $u \in C(E)$ , let*

$$\begin{aligned} F(t) &= \mathcal{E}(\Pi(a + tu)) \\ &= \int [\Pi(a + tu) - \Pi(a)][(dd^c\Pi(a + tu))^d + \dots + (dd^c\Pi(a))^d] \end{aligned}$$

and

$$G(t) := (d + 1) \int [\Pi(a + tu) - \Pi(a)](dd^c\Pi(a))^d.$$

Then

$$\lim_{t \rightarrow 0^+} \frac{F(t) - F(0)}{t} = \lim_{t \rightarrow 0^+} \frac{G(t) - G(0)}{t}.$$

*Proof.* Note that  $F(0) = \mathcal{E}(\Pi(a)) = 0$  and  $G(0) = 0$ . By concavity of  $\Pi$  (recall (2.12)) and linearity of  $f \rightarrow \int f(dd^c\Pi(a))^d$ , the function  $G(t)$  is concave so that

$$(4.9) \quad A := \lim_{t \rightarrow 0^+} \frac{G(t) - G(0)}{t}$$

exists. By concavity of  $\mathcal{E}$ , we have (recall (3.5))

$$\mathcal{E}(\Pi(a + tu)) \leq \mathcal{E}(\Pi(a)) + < \mathcal{E}'(\Pi(a)), \Pi(a + tu) - \Pi(a) >;$$



i.e., from (3.6) with  $u_1 = \Pi(a)$ ,  $u_2 = \Pi(a + tu)$  and  $v = \Pi(a)$ ,

$$\mathcal{E}(\Pi(a + tu)) \leq \mathcal{E}(\Pi(a)) + (d + 1) \int [\Pi(a + tu) - \Pi(a)](dd^c \Pi(a))^d.$$

Thus

$$\limsup_{t \rightarrow 0^+} \frac{F(t) - F(0)}{t} \leq A.$$

We prove

$$\liminf_{t \rightarrow 0^+} \frac{F(t) - F(0)}{t} \geq A.$$

Since  $A := \lim_{t \rightarrow 0^+} \frac{G(t) - G(0)}{t}$  exists, given  $\epsilon > 0$  we can choose  $\delta > 0$  sufficiently small so that

$$\frac{G(\delta) - G(0)}{\delta} = \frac{d + 1}{\delta} \int [\Pi(a + \delta u) - \Pi(a)](dd^c \Pi(a))^d \geq A - \epsilon;$$

i.e.,

$$(d + 1) \int [\Pi(a + \delta u) - \Pi(a)](dd^c \Pi(a))^d \geq \delta(A - \epsilon).$$

From Proposition 3.1, for  $t > 0$  sufficiently small we have

$$\begin{aligned} & \frac{\mathcal{E}(\Pi(a) + t[\Pi(a + \delta u) - \Pi(a)]) - \mathcal{E}(\Pi(a))}{t} \\ & \geq (d + 1) \int [\Pi(a + \delta u) - \Pi(a)](dd^c \Pi(a))^d - \delta\epsilon; \end{aligned}$$

i.e.,

$$\begin{aligned} \mathcal{E}((1 - t)\Pi(a) + t\Pi(a + \delta u)) &= \mathcal{E}(\Pi(a) + t[\Pi(a + \delta u) - \Pi(a)]) \\ &\geq \mathcal{E}(\Pi(a)) + t(d + 1) \int [\Pi(a + \delta u) - \Pi(a)](dd^c \Pi(a))^d - t\delta\epsilon. \end{aligned}$$

Combining these last two inequalities, we have

$$\mathcal{E}((1 - t)\Pi(a) + t\Pi(a + \delta u)) \geq \mathcal{E}(\Pi(a)) + t\delta A - 2t\delta\epsilon.$$

By concavity of  $\Pi$ ,

$$\Pi(a + t\delta u) = \Pi((1 - t)a + t(a + \delta u)) \geq (1 - t)\Pi(a) + t\Pi(a + \delta u)$$

so that, by monotonicity of  $\mathcal{E}$ ,

$$\mathcal{E}(\Pi(a + t\delta u)) \geq \mathcal{E}((1 - t)\Pi(a) + t\Pi(a + \delta u)) \geq \mathcal{E}(\Pi(a)) + t\delta A - 2t\delta\epsilon$$

for  $t > 0$  sufficiently small. Thus,

$$\liminf_{t \rightarrow 0^+} \frac{F(t) - F(0)}{t} \geq A - 2\epsilon$$

for all  $\epsilon > 0$ , yielding the result.  $\square$

We now finish the proof of Proposition 4.1 and Proposition 4.2 by finding  $A$  in (4.9). The proof that  $A = \int u(dd^c\Pi(a))^d$  was essentially given in the verification of (4.5) assuming (4.6) and (4.7); for the reader's convenience, we give the details. We write  $S_a := \text{supp}(dd^c\Pi(a))^d$ . For each  $t$ ,  $D(t) = \{z \in \mathbb{C}^d : \Pi(a + tu)(z) = a(z) + tu(z)\}$  is a bounded set. From Proposition 2.5,  $\Pi(a) = a$  a.e.  $-(dd^c\Pi(a))^d$  on  $S_a \subset D(0)$ ; thus

$$\begin{aligned}
& \int [\Pi(a + tu) - \Pi(a)](dd^c\Pi(a))^d = \int_{S_a} [\Pi(a + tu) - \Pi(a)](dd^c\Pi(a))^d \\
&= \int_{D(t) \cap S_a} [\Pi(a + tu) - \Pi(a)](dd^c\Pi(a))^d \\
&+ \int_{S_a \setminus D(t)} [\Pi(a + tu) - \Pi(a)](dd^c\Pi(a))^d \\
&= \int_{D(t) \cap S_a} [a + tu - a](dd^c\Pi(a))^d + \int_{S_a \setminus D(t)} [\Pi(a + tu) - \Pi(a)](dd^c\Pi(a))^d \\
&= \int_{D(t) \cap S_a} tu(dd^c\Pi(a))^d + \int_{S_a \setminus D(t)} [\Pi(a + tu) - \Pi(a)](dd^c\Pi(a))^d \\
&= \int_{S_a} tu(dd^c\Pi(a))^d + \int_{S_a \setminus D(t)} [\Pi(a + tu) - \Pi(a) - tu](dd^c\Pi(a))^d.
\end{aligned}$$

Now we use the observation (2.13) (or (2.14)) to see that

$$|\Pi(a + tu) - \Pi(a) - tu| = 0(t)$$

on the bounded set  $S_a \setminus D(t)$ ; the conclusion follows from Lemma 2.13.

We record an integrated version of Proposition 4.1 and Proposition 4.2 which we will use.

**Proposition 4.4.** *For admissible weights  $a, b \in C^2(E)$  on an unbounded closed set  $E$  satisfying (2.15),*

$$(4.10) \quad \mathcal{E}(\Pi(b), \Pi(a)) = (d+1) \int_{t=0}^1 dt \int_{\mathbb{C}^d} (b-a)(dd^c\Pi(a + t(b-a)))^d;$$

and for a compact set  $K$  with admissible weight  $a$  and  $u \in C^2(K)$ ,

$$(4.11) \quad \mathcal{E}(\Pi(a + u), \Pi(a)) = (d+1) \int_{t=0}^1 dt \int_{\mathbb{C}^d} u(dd^c\Pi(a + tu))^d.$$

*Proof.* We prove (4.10) as (4.11) is similar. We begin with Proposition 4.1 using  $v = \Pi(a)$  so that  $F(t) = \mathcal{E}(\Pi(a + t(b - a)), \Pi(a))$  and (4.2) becomes

$$F'(t) = (d + 1) \int_{\mathbb{C}^d} (b - a)(dd^c \Pi(a + t(b - a)))^d.$$

Integrating this expression from  $t = 0$  to  $t = 1$  gives (4.10) since  $F(1) - F(0) = \mathcal{E}(\Pi(b), \Pi(a))$ .  $\square$

## 5. THE MAIN THEOREM.

In this section, we state and prove the main result which relates asymptotics of certain ball-volume ratios with energies associated with  $P$ -extremal functions. For  $E \subset \mathbb{C}^d$  closed, following notation in [4], we let  $\phi$  be an admissible weight on  $E$ . Let

$$\mathcal{B}^\infty(E, n\phi) := \{p_n \in \text{Poly}(nP) : |p_n(z)|^2 e^{-2n\phi(z)} \leq 1 \text{ on } E\}$$

be an  $L^\infty$ -ball and, if  $\mu$  is a measure on  $E$ , let

$$\mathcal{B}^2(E, \mu, n\phi) := \{p_n \in \text{Poly}(nP) : \int_E |p_n|^2 e^{-2n\phi} d\mu \leq 1\}$$

be an  $L^2$ -ball in  $\text{Poly}(nP)$ . The key result is the following.

**Theorem 5.1.** *Given  $\phi, \phi'$  admissible weights on  $E, E'$ ,*

$$\lim_{n \rightarrow \infty} \frac{-(d+1)n_d}{2nd_n} [\mathcal{B}^\infty(E, n\phi) : \mathcal{B}^\infty(E', n\phi')] = \mathcal{E}(V_{P,E,\phi}^*, V_{P,E',\phi'}^*).$$

*If  $\mu, \mu'$  are measures on  $E, E'$  where  $\mu$  is a Bernstein-Markov measure for  $(P, E, \phi)$  and  $\mu'$  is a Bernstein-Markov measure for  $(P, E', \phi')$ , then*

$$\lim_{n \rightarrow \infty} \frac{-(d+1)n_d}{2nd_n} [\mathcal{B}^2(E, \mu, n\phi) : \mathcal{B}^2(E', \mu', n\phi')] = \mathcal{E}(V_{P,E,\phi}^*, V_{P,E',\phi'}^*).$$

**Remark 5.2.** Taking  $E' = T$  and  $\phi' = 0$ , from (2.7) we have  $V_{P,E',\phi'}^* = H_P$ . Now taking  $\mu' = \mu_T$  and taking  $(K, \mu, Q)$  for the triple  $(E, \mu, \phi)$  where  $K$  is compact and  $\mu$  is a Bernstein-Markov measure for  $(P, K, Q)$ , we verify Conjecture 2.20. We use (2.35) and (2.25) to obtain (2.37), the existence of the limit

$$(5.1) \quad \lim_{n \rightarrow \infty} \frac{1}{2l_n} \log \det(G_n^{\mu,w}) = \frac{-1}{n_d d \mathcal{A}} \mathcal{E}(V_{P,K,Q}^*, H_P) = \log \mathcal{F}_P(K, Q).$$

Thus we obtain the asymptotics of weighted Gram determinants associated to  $(K, \mu, Q)$  as well as the other results mentioned in Section 2: the existence of the limit of the scaled maximal weighted Vandermondes

$$\delta^w(K) := \lim_{n \rightarrow \infty} \delta_n^w(K) = \mathcal{F}_P(K, Q)$$

in (2.22) and Proposition 2.21 on  $P$ -optimal measures.

The first step of the proof is a version of Bergman asymptotics in a special case.

**5.1. Weighted Bergman asymptotics in  $\mathbb{C}^d$ .** We state a result on Bergman asymptotics in [3]. The setting is this:  $\phi \in C^{1,1}(\mathbb{C}^d)$  with

$$(5.2) \quad \phi(z) \geq (1 + \epsilon)H_P(z) \text{ for } |z| \gg 1 \text{ for some } \epsilon > 0.$$

We will call a global admissible weight  $\phi$  satisfying (5.2) *strongly admissible*. For  $p_n \in \text{Poly}(nP)$ , we write

$$\|p_n\|_{n\phi}^2 := \|p_n\|_{\omega_d, n\phi}^2 = \int_{\mathbb{C}^d} |p_n(z)|^2 e^{-2n\phi(z)} \omega_d(z)$$

where  $\omega_d$  is Lebesgue measure on  $\mathbb{C}^d$ . Using (2.2), under the growth assumption on  $\phi$ , if  $n > \frac{d}{\epsilon k A}$  where  $P \subset A\Sigma$  then for each polynomial  $p_n \in \text{Poly}(nP)$ ,  $\|p_n\|_{n\phi} < +\infty$ .

Given an orthonormal basis  $\{q_1, \dots, q_{d_n}\}$  of  $\text{Poly}(nP)$ , in this section we use the notation

$$B_{n,\phi}(z) := \left[ \sum_{j=1}^{d_n} |q_j(z)|^2 \right] e^{-2n\phi(z)}$$

for the  $n$ -th Bergman function; and we recall that

$$B_{n,\phi}(z) = \sup_{p_n \in \text{Poly}(nP) \setminus \{0\}} |p_n(z)|^2 e^{-2n\phi(z)} / \|p_n\|_{n\phi}^2.$$

Finally, let

$$S := \{z \in \mathbb{C}^d : dd^c \phi(z) \text{ exists and } dd^c \phi(z) > 0\}$$

and if  $u$  is a  $C^{1,1}$  function such that  $(dd^c u)^d$  is absolutely continuous with respect to Lebesgue measure, we write

$$\det(dd^c u) \omega_d := (dd^c u)^d.$$

**Theorem 5.3.** *Given  $\phi \in C^{1,1}(\mathbb{C}^d)$  satisfying (5.2), we have the following:  $V_{P,\mathbb{C}^d,\phi} \in C^{1,1}(\mathbb{C}^d)$ ;  $(dd^c V_{P,\mathbb{C}^d,\phi})^d$  has compact support and is absolutely continuous with respect to Lebesgue measure;*

$$(dd^c V_{P,\mathbb{C}^d,\phi})^d = \det(dd^c V_{P,\mathbb{C}^d,\phi}) \omega_d$$

*as  $(d, d)$ -forms with  $L_{loc}^\infty(\mathbb{C}^d)$  coefficients; and a.e. on the set  $D := \{V_{P,\mathbb{C}^d,\phi} = \phi\}$  we have  $\det(dd^c \phi) = \det(dd^c V_{P,\mathbb{C}^d,\phi})$ . Moreover,*

$$\frac{n_d}{d_n} B_{n,\phi} \rightarrow \chi_{D \cap S} \det(dd^c \phi) \text{ in } L^1(\mathbb{C}^d)$$

*and the measures*

$$\frac{n_d}{d_n} B_{n,\phi} \omega_d \rightarrow (dd^c V_{P,\mathbb{C}^d,\phi})^d \text{ weakly.}$$

Recall the (strong) admissibility of  $\phi$  implies, by Proposition 2.5, that  $(dd^c V_{P,\mathbb{C}^d,\phi})^d$  has compact support.

**Remark 5.4.** From [6],  $(D, \omega_d|_D, \phi|_D)$  satisfies a weighted Bernstein-Markov property for  $\mathcal{P}_n$  or  $A\mathcal{P}_n$ ; from Remark 2.10,  $\omega_d|_D$  is a Bernstein-Markov measure for the triple  $(P, D, \phi)$ . Using Proposition 2.9,

$$\sup_{\mathbb{C}^d} |p_n e^{-n\phi}| = \sup_D |p_n e^{-n\phi}|$$

for  $p_n \in \text{Poly}(nP)$ . Hence, from (2.9),

$$\sup_{\mathbb{C}^d} |p_n e^{-n\phi}| \leq M_n \left[ \int_D |p_n|^2 e^{-2n\phi} \omega_d \right]^{1/2} \leq M_n \left[ \int_{\mathbb{C}^d} |p_n|^2 e^{-2n\phi} \omega_d \right]^{1/2}$$

where  $M_n^{1/n} \rightarrow 1$ . This last integral is finite by (2.2).

**5.2. Proof of the Main Theorem.** We consider several cases.

*Case 1:  $E = E' = \mathbb{C}^d$  and  $\phi, \phi' \in C^2(\mathbb{C}^d)$  strongly admissible with  $\phi' = \phi$  outside a ball  $\mathcal{B}_R$  for some  $R$ ;  $d\mu = d\mu' = \omega_d$ :*

We begin in the  $L^2$ -Case 1. Note that (2.15) holds for then all of the weights  $\phi + t(\phi' - \phi)$  are strongly admissible with a uniform  $\epsilon$  (recall (5.2)). Let  $u := \phi' - \phi$ ; then  $u$  is continuous with compact support. For  $0 \leq t \leq 1$  let

$$\phi_t := \phi + tu = \phi + t(\phi' - \phi) = (1-t)\phi + t\phi'$$

so that  $\phi_0 = \phi$  and  $\phi_1 = \phi'$ ; equivalently,  $w_t(z) := w(z) \exp(-tu(z))$  (note  $w_0 = w = e^{-\phi}$  and  $w_1 = w' = e^{-\phi'}$ ). Then from Theorem 5.3, for

each  $t$ ,

$$\frac{n_d}{d_n} B_{n,\phi+tu} \cdot \omega_d \rightarrow (dd^c \Pi(\phi + tu))^d \text{ weakly.}$$

Now set

$$f_n(t) := -\frac{1}{2l_n} \log \det(G_n^{\mu, w_t}(\beta_n))$$

where  $\mu = \mu_n := \omega_d$  for all  $n$  and the basis  $\beta_n := \{p_1, \dots, p_{d_n}\}$  of  $Poly(nP)$  is chosen to be an orthonormal basis with respect to the weighted  $L^2$ -norm  $p \rightarrow \|w^n p\|_{L^2(\mu)}$ . Then  $G_n^{\mu, w}(\beta_n)$  is the  $d_n \times d_n$  identity matrix so that we have  $f_n(0) = 0$ ; and, using Lemma 2.17 and the fact that  $u$  has compact support (thus all weights  $w_t$  are admissible),

$$\lim_{n \rightarrow \infty} \frac{l_n}{nd_n} f'_n(t) = \lim_{n \rightarrow \infty} \frac{1}{d_n} \int u B_{n,\phi+tu} \omega_d = \frac{1}{n_d} \int u (dd^c \Pi(\phi + tu))^d.$$

We now integrate  $\frac{l_n}{nd_n} f'_n(t)$  from  $t = 0$  to  $t = 1$ :

$$\begin{aligned} \frac{l_n}{nd_n} [f_n(1) - f_n(0)] &= \frac{l_n}{nd_n} [f_n(1)] = \frac{-1}{2nd_n} \log \det(G_n^{\mu, w'}(\beta_n)) \\ &= \frac{-1}{2nd_n} [\mathcal{B}^2(\mathbb{C}^d, \mu, n\phi) : \mathcal{B}^2(\mathbb{C}^d, \mu, n\phi')] \text{ (from (2.34))} \\ &= \frac{1}{d_n} \int_{t=0}^1 dt \int B_{n,\phi+tu}(\phi - \phi') \omega_d \text{ (from Lemma 2.17)} \\ &\rightarrow \frac{1}{n_d} \int_{t=0}^1 dt \int (\phi - \phi')(dd^c \Pi(\phi + tu))^d. \end{aligned}$$

But by (4.10), since (2.15) holds,

$$(d+1) \int_{t=0}^1 dt \int (\phi - \phi')(dd^c \Pi(\phi + tu))^d = \mathcal{E}(\Pi(\phi'), \Pi(\phi))$$

which proves Theorem 5.1 in  $L^2$ -Case 1. By Remark 5.4 this also proves the  $L^\infty$ -Case 1.

*Case 2:*  $E = E' = \mathbb{C}^d$  and  $\phi, \phi' \in C^2(\mathbb{C}^d)$  strongly admissible;  $d\mu = d\mu' = \omega_d$ :

We first do the  $L^\infty$ -Case 2. Remark 2.6 and Proposition 2.3 imply that

$$\Pi(\phi) = \Pi_{S_w}(\phi|_{S_w})$$

where  $S_w = \text{supp}(dd^c \Pi(\phi))^d$  is compact; moreover, for  $p_n \in Poly(nP)$ , from Proposition 2.9,  $\|p_n e^{-n\phi}\|_{S_w} = \|p_n e^{-n\phi}\|_{\mathbb{C}^d}$  so that

$$\mathcal{B}^\infty(S_w, n\phi|_{S_w}) = \mathcal{B}^\infty(\mathbb{C}^d, n\phi).$$

Thus modifying  $\phi, \phi'$  outside a large ball in such a way to make them equal outside a perhaps larger ball, we neither change the  $L^\infty$ -ball volume ratios nor the  $P$ -extremal functions  $\Pi(\phi), \Pi(\phi')$ . Hence the  $L^\infty$ -Case 2 follows from the  $L^\infty$ -Case 1. By Remark 5.4 this also proves the  $L^2$ -Case 2.

*Case 3 (general):*  $E, E' \subset \mathbb{C}^d$  closed with admissible weights  $\phi, \phi'; \mu, \mu'$  Bernstein-Markov measures for  $(P, E, \phi), (P, E', \phi')$ :

We consider the  $L^\infty$ -Case 3 only; the  $L^2$ -Case 3 follows from the definition of Bernstein-Markov measure for  $(P, E, \phi), (P, E', \phi')$ . We claim that by the cocycle property for the ball volume ratios  $[A : B]$  and energies  $\mathcal{E}(u_1, u_2)$ , we may assume that one of the sets is  $\mathbb{C}^d$  with a strongly admissible  $C^2(\mathbb{C}^d)$  weight  $\widehat{\phi}$ . For, using the notation  $\Pi_E(\phi) := V_{P,E,\phi}^*$ , we have

$$\mathcal{E}(\Pi_E(\phi), \Pi_{E'}(\phi')) = -\mathcal{E}(\Pi_{E'}(\phi'), \Pi_{\mathbb{C}^d}(\widehat{\phi})) + \mathcal{E}(\Pi_E(\phi), \Pi_{\mathbb{C}^d}(\widehat{\phi})).$$

Both terms on the right have the second term being  $\Pi_{\mathbb{C}^d}(\widehat{\phi})$ . Similarly, with respect to the ball volume ratios, for each  $n$  we have

$$\begin{aligned} & [\mathcal{B}^\infty(E, n\phi) : \mathcal{B}^\infty(E', n\phi')] \\ &= -[\mathcal{B}^\infty(E', n\phi') : \mathcal{B}^\infty(\mathbb{C}^d, n\widehat{\phi})] + [\mathcal{B}^\infty(E, n\phi) : \mathcal{B}^\infty(\mathbb{C}^d, n\widehat{\phi})]. \end{aligned}$$

Now to deduce the case where one of the sets is  $\mathbb{C}^d$  with a strongly admissible  $C^2(\mathbb{C}^d)$  weight  $\widehat{\phi}$  and the other is a general closed set  $E$  with admissible weight  $\phi$  from Case 2 where both sets are  $\mathbb{C}^d$  with strongly admissible  $C^2(\mathbb{C}^d)$  weights  $\widehat{\phi}, \psi$ , we first observe that we may assume  $E$  is compact (i.e., bounded). For recall again from Proposition 2.3 that if  $w = e^{-\phi}$ ,  $\Pi_E(\phi) = \Pi_{S_w}(\phi|_{S_w})$  where  $S_w = \text{supp}(dd^c \Pi_E(\phi))^d$  is compact; and for  $p_n \in \text{Poly}(nP)$ ,  $\|p_n e^{-n\phi}\|_{S_w} = \|p_n e^{-n\phi}\|_E$  so that

$$\mathcal{B}^\infty(S_w, n\phi|_{S_w}) = \mathcal{B}^\infty(E, n\phi).$$

Thus we assume  $E$  is compact; since  $V_{P,E,\phi}^* \in L_{P,+}$ , we can also assume  $\phi$  is bounded above on  $E$ . We take a large sublevel set  $B_R := \{z \in \mathbb{C}^d : H_P(z) < \log R\}$  containing  $E$  and extend  $\phi$  from  $E$  to  $\widehat{\psi}$  on  $\mathbb{C}^d$ :

$$\widehat{\psi} := \phi \text{ on } E; \quad \widehat{\psi} = 2 \log R \text{ on } B_R \setminus E; \quad \widehat{\psi} = 2kH_P(z) \text{ on } \mathbb{C}^d \setminus B_R.$$

We have  $\widehat{\psi}$  is lowersemicontinuous and by taking  $R$  sufficiently big  $\Pi_{\mathbb{C}^d}(\widehat{\psi}) = \Pi_E(\phi)$ ; then we take a sequence of strongly admissible  $C^2(\mathbb{C}^d)$

weights  $\{\phi_j\}$  with  $\phi_j \uparrow \widehat{\psi}$ . We can apply Case 2 to  $(\mathbb{C}^d, \phi_j)$  and  $(\mathbb{C}^d, \widehat{\phi})$  to conclude

$$\lim_{n \rightarrow \infty} \frac{-(d+1)n_d}{2nd_n} [\mathcal{B}^\infty(\mathbb{C}^d, n\phi_j) : \mathcal{B}^\infty(\mathbb{C}^d, n\widehat{\phi})] = \mathcal{E}(\Pi_{\mathbb{C}^d}(\phi_j), \Pi_{\mathbb{C}^d}(\widehat{\phi})).$$

But  $\phi_j \uparrow \widehat{\psi}$  implies  $\Pi_{\mathbb{C}^d}(\phi_j) \uparrow \Pi_{\mathbb{C}^d}(\widehat{\psi}) = \Pi_E(\phi)$  and hence

$$(5.3) \quad \mathcal{E}(\Pi_{\mathbb{C}^d}(\phi_j), \Pi_{\mathbb{C}^d}(\widehat{\phi})) \text{ converges to } \mathcal{E}(\Pi_E(\phi), \Pi_{\mathbb{C}^d}(\widehat{\phi}))$$

as  $j \rightarrow \infty$  by Lemma 3.5.

We want to conclude that

$$(5.4) \quad \lim_{n \rightarrow \infty} \frac{-(d+1)n_d}{2nd_n} [\mathcal{B}^\infty(E, n\phi) : \mathcal{B}^\infty(\mathbb{C}^d, n\widehat{\phi})] = \mathcal{E}(\Pi_E(\phi), \Pi_{\mathbb{C}^d}(\widehat{\phi})).$$

To this end, first observe that

$$\begin{aligned} -\mathcal{E}(\Pi_{\mathbb{C}^d}(\phi_j), \Pi_{\mathbb{C}^d}(\widehat{\phi})) &= \lim_{n \rightarrow \infty} \frac{(d+1)n_d}{2nd_n} [\mathcal{B}^\infty(\mathbb{C}^d, n\phi_j) : \mathcal{B}^\infty(\mathbb{C}^d, n\widehat{\phi})] \\ &\leq \liminf_{n \rightarrow \infty} \frac{(d+1)n_d}{2nd_n} [\mathcal{B}^\infty(E, n\phi) : \mathcal{B}^\infty(\mathbb{C}^d, n\widehat{\phi})] \\ &\leq \limsup_{n \rightarrow \infty} \frac{(d+1)n_d}{2nd_n} [\mathcal{B}^\infty(E, n\phi) : \mathcal{B}^\infty(\mathbb{C}^d, n\widehat{\phi})] \end{aligned}$$

since  $\Pi_{\mathbb{C}^d}(\phi_j) \uparrow \Pi_E(\phi)$  implies from (2.39) that

$$[\mathcal{B}^\infty(\mathbb{C}^d, n\phi_j) : \mathcal{B}^\infty(\mathbb{C}^d, n\widehat{\phi})] \leq [\mathcal{B}^\infty(E, n\phi) : \mathcal{B}^\infty(\mathbb{C}^d, n\widehat{\phi})].$$

Now we take a sequence of smooth, strongly admissible weights  $\{\psi_j\}$  on  $\mathbb{C}^d$  with  $\psi_j \downarrow \Pi_E(\phi)$ ; e.g., we may take  $\psi_j = (1 + \epsilon_j)[(\Pi_E(\phi))_{\epsilon_j}]$  where  $(\Pi_E(\phi))_{\epsilon_j}$  is a smoothing of  $\Pi_E(\phi)$ . Then  $\Pi_{\mathbb{C}^d}(\psi_j) \downarrow \Pi_E(\phi)$  and

$$\begin{aligned} &\limsup_{n \rightarrow \infty} \frac{(d+1)n_d}{2nd_n} [\mathcal{B}^\infty(E, n\phi) : \mathcal{B}^\infty(\mathbb{C}^d, n\widehat{\phi})] \\ &\leq \lim_{n \rightarrow \infty} \frac{(d+1)n_d}{2nd_n} [\mathcal{B}^\infty(\mathbb{C}^d, n\psi_j) : \mathcal{B}^\infty(\mathbb{C}^d, n\widehat{\phi})] \end{aligned}$$

again by (2.39); this limit equals

$$-\mathcal{E}(\Pi_{\mathbb{C}^d}(\psi_j), \Pi_{\mathbb{C}^d}(\widehat{\phi}))$$

by applying Case 2, this time to  $(\mathbb{C}^d, \psi_j)$  and  $(\mathbb{C}^d, \widehat{\phi})$ . Now

$$(5.5) \quad \mathcal{E}(\Pi_{\mathbb{C}^d}(\psi_j), \Pi_{\mathbb{C}^d}(\widehat{\phi})) \text{ converges to } \mathcal{E}(\Pi_E(\phi), \Pi_{\mathbb{C}^d}(\widehat{\phi}))$$

as  $j \rightarrow \infty$  by (3.7). Then (5.3) and (5.5) imply (5.4) which completes the proof of Theorem 5.1.



6. ASYMPTOTIC WEIGHTED  $P$ -FEKETE MEASURES, WEIGHTED  $P$ -OPTIMAL MEASURES AND BERGMAN ASYMPTOTICS.

As in [5], we will apply the following calculus lemma (cf., Lemma 7.6 in [4] or Lemma 3.1 in [5]) to an appropriate sequence of real-valued functions  $\{f_n\}$  in order to prove a general result, Proposition 6.2, on convergence to the Monge-Ampère measure of a weighted  $P$ -extremal function. This proposition utilizes the differentiability result, Proposition 4.2, and yields immediate corollaries on the items in the title of this section.

**Lemma 6.1.** *Let  $f_n$  be a sequence of real-valued, concave functions on  $\mathbb{R}$  and let  $g$  be a function on  $\mathbb{R}$ . Suppose*

$$\liminf_{n \rightarrow \infty} f_n(t) \geq g(t) \text{ for all } t \text{ and } \lim_{n \rightarrow \infty} f_n(0) = g(0)$$

*and that  $f_n$  and  $g$  are differentiable at 0. Then  $\lim_{n \rightarrow \infty} f'_n(0) = g'(0)$ .*

Here “differentiable at the origin” means that the usual (two-sided) limit of the difference quotients exists; the conclusion is not true with one-sided limits.

As in Lemma 2.17 in subsection 2.4, given a closed set  $E$ , an admissible weight  $w = e^{-Q}$  on  $E$ , and a function  $u \in C(E)$ , we consider the weight  $w_t(z) := w(z) \exp(-tu(z))$ ,  $t \in \mathbb{R}$ , and we let  $\{\mu_n\}$  be a sequence of measures on  $E$ .

*For the rest of this section, we take  $E = K$ , a compact set, so each  $w_t$  is admissible. In addition, in computing Gram matrices, we fix the standard monomial basis  $\beta_n = \{e_1, \dots, e_{d_n}\}$  of  $\text{Poly}(nP)$ ; and we fix  $v = H_P$  in the second slot of  $\mathcal{E}(u, v)$ .*

Now let  $\mu$  be a probability measure on  $K$  and let  $u \in C^2(K)$ . Recalling (2.25), define

$$g(t) := -\log \delta^{w_t}(K) = \frac{1}{n_d d \mathcal{A}} \mathcal{E}(\Pi(Q + tu)).$$

Then

$$g(0) = -\log \delta^w(K) = \frac{1}{n_d d \mathcal{A}} \mathcal{E}(\Pi(Q)).$$

From Proposition 4.2

$$g'(0) = \frac{d+1}{n_d d \mathcal{A}} \int_K u(z) (dd^c \Pi(Q))^d.$$

Note that for each  $n$ ,  $\mu_n$  is a candidate to be a  $P$ -optimal measure of order  $n$  for  $K$  and  $w_t$ . Thus, if  $\mu_n^t$  is a  $P$ -optimal measure of order  $n$  for  $K$  and  $w_t$ , we have

$$\det G_n^{\mu_n, w_t} \leq \det G_n^{\mu_n^t, w_t}$$

and, from Proposition 2.21 (see Remark 5.2),

$$\lim_{n \rightarrow \infty} \frac{1}{2l_n} \cdot \log \det G_n^{\mu_n^t, w_t} = \log \delta^{w_t}(K) = -g(t).$$

Thus with

$$f_n(t) := -\frac{1}{2l_n} \log \det(G_n^{\mu_n, w_t})$$

as in (2.31), we have

$$f_n(0) = \frac{-1}{2l_n} \log \det(G_n^{\mu_n, w}) \text{ and } \liminf f_n(t) \geq g(t) \text{ for all } t.$$

From Lemma 2.17, we have

$$f'_n(0) = \frac{n}{l_n} \int_K u(z) B_n^{\mu_n, w}(z) d\mu_n$$

and from Lemma 2.18, the functions  $f_n(t)$  are concave, i.e.,  $f''_n(t) \leq 0$ .

Using Lemma 6.1 and (2.25), we have the following general result.

**Proposition 6.2.** *Let  $K \subset \mathbb{C}^d$  be compact with admissible weight  $w$ . Let  $\{\mu_n\}$  be a sequence of probability measures on  $K$  with the property that*

$$(6.1) \quad \lim_{n \rightarrow \infty} \frac{1}{2l_n} \log \det(G_n^{\mu_n, w}) = \log \mathcal{F}_P(K, Q)$$

*i.e.,  $\lim_{n \rightarrow \infty} f_n(0) = g(0)$ . Then*

$$\frac{n}{l_n} B_n^{\mu_n, w} d\mu_n \rightarrow \frac{d+1}{n_d d \mathcal{A}} (dd^c \Pi(Q))^d \text{ weak-}^*; \text{ i.e.,}$$

$$(6.2) \quad \frac{n_d}{d_n} B_n^{\mu_n, w} d\mu_n \rightarrow (dd^c \Pi(Q))^d \text{ weak-}^*.$$

Note that since all  $\mu_n$  are probability measures on  $K$ , to verify weak- $^*$  convergence, it suffices to test with  $C^2$ -functions on  $K$ .

From Theorem 5.1 (more precisely, Remark 5.2 and equation (5.1)) we have the general Bergman asymptotic result.

**Corollary 6.3. [Bergman Asymptotics]** *If  $\mu$  is a Bernstein-Markov measure for the triple  $(P, K, Q)$ , then*

$$\frac{n_d}{d_n} B_n^{\mu, w} d\mu \rightarrow (dd^c \Pi(Q))^d \text{ weak-}^*.$$

Next, suppose  $\mu_n$  is a  $P$ -optimal measure of order  $n$  for  $K$  and  $w$ .

**Corollary 6.4. [Weighted Optimal Measures]** *Let  $K \subset \mathbb{C}^d$  be compact with admissible weight  $w$ . Let  $\{\mu_n\}$  be a sequence of  $P$ -optimal measures for  $K, w$ . Then*

$$\mu_n \rightarrow \frac{1}{n_d} (dd^c \Pi(Q))^d \text{ weak-}^*.$$

*Proof.* We have  $B_n^{\mu_n, w} = d_n$  a.e.  $\mu_n$  on  $K$  from 2.33 so that the result follows immediately from Proposition 2.21 and Proposition 6.2, specifically, equation (6.2).  $\square$

Finally, we prove the result promised in Section 2.

**Corollary 6.5. [Asymptotic Weighted  $P$ -Fekete Points]** *Let  $K \subset \mathbb{C}^d$  be compact with admissible weight  $w$ . For each  $n$ , take points  $z_1^{(n)}, z_2^{(n)}, \dots, z_{d_n}^{(n)} \in K$  for which*

$$(6.3) \quad \lim_{n \rightarrow \infty} [|VDM(z_1^{(n)}, \dots, z_{d_n}^{(n)})| w(z_1^{(n)})^n \dots w(z_{d_n}^{(n)})^n]^{\frac{1}{n}} = \mathcal{F}_P(K, Q)$$

*(asymptotically weighted  $P$ -Fekete points) and let  $\mu_n := \frac{1}{d_n} \sum_{j=1}^{d_n} \delta_{z_j^{(n)}}$ . Then*

$$\mu_n \rightarrow \frac{1}{n_d} (dd^c \Pi(Q))^d \text{ weak-}^*.$$

*Proof.* By direct calculation, we have  $B_n^{\mu_n, w}(z_j^{(n)}) = d_n$  for  $j = 1, \dots, d_n$  and hence a.e.  $\mu_n$  on  $K$ . Indeed, this property holds for *any* discrete, equally weighted measure  $\mu_n := \frac{1}{d_n} \sum_{j=1}^{d_n} \delta_{z_j^{(n)}}$  with

$$|VDM(z_1^{(n)}, \dots, z_{d_n}^{(n)})| w(z_1^{(n)})^n \dots w(z_{d_n}^{(n)})^n \neq 0.$$

Using

$$\det(G_n^{\mu_n, w}) = \frac{1}{d_n^{d_n}} |VDM(z_1^{(n)}, \dots, z_{d_n}^{(n)})|^2 w(z_1^{(n)})^{2n} \dots w(z_{d_n}^{(n)})^{2n},$$

the result follows from Proposition 6.2, specifically, equation (6.2).  $\square$

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